MATH 2050C Lecture 3 (Jan 18)

[[Midterm confirmed: Mar 3, 8:30-10:00 am, in-class.]]

Goal: $\mathbb{R}$ is a complete ordered field.
today

Def^n: (Absolute value) Let $a \in \mathbb{R}$.

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Note: $|a| \geq 0 \quad \forall a \in \mathbb{R}$

Prop: (a) $|ab| = |a| \cdot |b|$

(b) $|a|^2 = a^2$

* (c) Let $c \geq 0$. Then $|a| \leq c \iff -c \leq a \leq c$

(d) $-|a| \leq a \leq |a|$

Proof: (a) We exhaust all possible cases from Trichotomy (O2).

Case 1: Either a or b is 0.

Then, $ab = 0 \Rightarrow |ab| = 0$.

Also, if $a = 0$, then $|a| = 0 \Rightarrow |a| \cdot |b| = 0$.

Same for $b = 0$. So, $|ab| = |a||b| = 0$.

Case 2: $a > 0$ and $b < 0$.

Then, by Prop. last time, $ab < 0 \Rightarrow |ab| = -ab$.

Also, $a > 0 \Rightarrow |a| = a$
$b < 0 \Rightarrow |b| = -b$
$\Big\}\ |a| \cdot |b| = a \cdot (-b) = -ab$. // Same

Ex: Check this!

Case 3: $a > 0$ and $b > 0$
Case 4: $a < 0$ and $b < 0$ $\Big\}$ Ex:

Same as Case 2 → Case 5: $a < 0$ and $b > 0$

(b) Take $b = a$ in (a),

$$a^2 = |a^2| = |ab| = |a||b| = |a|\cdot|a| = |a|^2.$$

$\because a^2 \geq 0 \quad \forall a \in \mathbb{R}.$

(c) Exhaust all cases of a by trichotomy (Ex:)

(d) Follows from (c) by taking $C = |a| \geq 0$. ∎

Some Useful Inequalities

(1) AM-GM inequality : $\underbrace{\sqrt{ab}}_{\geq 0} \leq \frac{1}{2}(a+b) \quad \forall a, b \geq 0$

(2) Triangle inequality : $|a+b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$

(3) Bernoulli's inequality : $\underbrace{(1+x)^n}_{\geq 0} \geq 1 + n\cdot x \quad \forall x > -1, \forall n \in \mathbb{N}$

Proof: (1) Let $a, b \geq 0$, then $\sqrt{a}, \sqrt{b}$ exist (Assume this).

By previous lemma,

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = \underbrace{(\sqrt{a})^2}_{\text{// def}^{\underline{n}}} - 2\sqrt{a}\sqrt{b} + \underbrace{(\sqrt{b})^2}_{\text{// def}^{\underline{n}}}$$
$$= a - 2\sqrt{a}\sqrt{b} + b$$

Rearranging gives the desired inequality.

(2) By (d) above, we have

$$\left.\begin{array}{l} -|a| \leq a \leq |a| \\ -|b| \leq b \leq |b| \end{array}\right\} \overset{\text{add}}{\Longrightarrow} -(|a|+|b|) \leq a+b \leq |a|+|b|$$
$$\overset{(c)}{\Longrightarrow} |a+b| \leq |a| + |b|.$$

(3) Induction on n.

$\underline{n=1}$: Trivial since $(1+x)^n = 1+x = 1+n\cdot x$. when $n=1$.

Assume $\underline{n=k \text{ is true}}$, then for $n = k+1$,

$$(1+x)^{k+1} = \underbrace{(1+x)}_{\substack{\vee \\ 0 \\ \because x > -1}} \underbrace{(1+x)^k}_{\substack{\text{IV} \\ n=k}}$$

$$
\begin{aligned}
&\geq (1+x)(1+k\cdot x) && \left(\because n=k \text{ is true and } x>-1\right)\\
&= 1+(k+1)x + k\cdot x^2 \\
&\geq 1+(k+1)x && \left(\because k>0,\ x^2\geq 0\right)
\end{aligned}
$$

By M.I., we are done. $\square$

Remark: Let $a, b \geq 0$. Then

$$a \leq b \iff a^2 \leq b^2 \iff \sqrt{a} \leq \sqrt{b}.$$

Prop: (Reversed Triangle Ineq.)

$$\big|\,|a|-|b|\,\big| \leq |a-b| \qquad \forall\, a, b \in \mathbb{R}.$$

Pf: Tutorial.

Def$^{\underline{n}}$ / Thm (Completeness Property of $\mathbb{R}$)

Every $\phi \neq S \subseteq \mathbb{R}$ that has an "upper bound" ?

must have a "supremum" in $\mathbb{R}$.

?

We first make sense of the ?'s.

Def$^{\underline{n}}$: Let $\phi \neq S \subseteq \mathbb{R}$.

(a) S is bounded above if $\exists\, u \in \mathbb{R}$ s.t. $s \leq u \quad \forall\, s \in S$.

Any such $u \in \mathbb{R}$ is called an upper bound of S.

(b) S is bounded below if $\exists\, w \in \mathbb{R}$ s.t. $s \geq w \quad \forall\, s \in S$.

Any such $w \in \mathbb{R}$ is called a lower bound of S.

(c) S is bounded if it is both bdd above AND below.

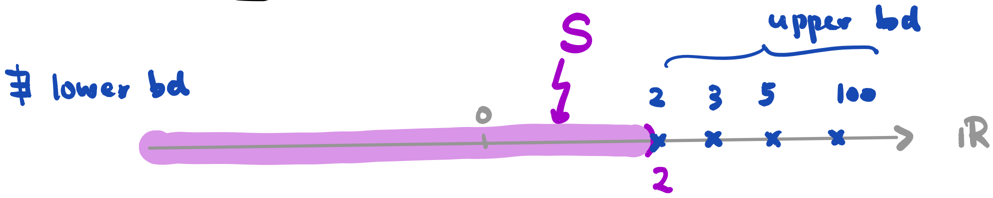
Otherwise, S is unbounded.

Example: $S := \{ x \in \mathbb{R} \mid x < 2 \}$

Note: There are many upper bds, e.g. $2, 3, 5, 100, \sqrt{100}$ etc...

$\Rightarrow$ S is bdd above.

But S is NOT bdd below. (Ex: prove it)



Def$^{\underline{n}}$: Let $\phi \neq S \subseteq \mathbb{R}$.

(a) Suppose S is bdd above.

Then, $u \in \mathbb{R}$ is called a supremum (or least upper bound) of S if the following holds:

[**Notation:** $u = \sup S$ or $\text{l.u.b.}\, S$]

(i) u is an upper bd of S

(ii) $u \leq v$ for any upper bd v of S

(b) Similarly, we can define infimum (or greatest lower bound)

[**Notation:** $\inf S$ or $\text{g.l.b.}\, S$] Ex: Write this down.

Lemma: $\sup S$, if exists, is unique.

Proof: Suppose there are two $u, w \in \mathbb{R}$ which are supremum of S

Therefore, u, w satisfy (i), (ii) in the def$^{\underline{n}}$ above.

By (i) for w and (ii) for u, we have

$$u \leq w$$ $\because$ w is an upper bd

Similarly, by (i) for u and (ii) for w, we have

$$w \le u \quad \leftarrow \because u \text{ is an upper bd.}$$

Thus, $u = w$. $\square$

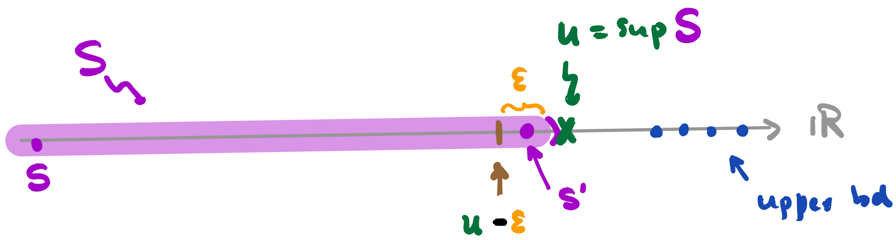
Prop: Let $\phi \neq S \subseteq \mathbb{R}$. Then $u = \sup S$ iff

(i) $s \le u \quad \forall s \in S$

(ii) $\forall \varepsilon > 0$, $\exists s' \in S$ s.t. $u - \varepsilon < s'$

$\leftarrow$ (Useful way to prove $u = \sup S$)

Picture:



Proof: "$\Rightarrow$" Suppose $u = \sup S$.

By (i), u is an upper bd of S

$\Rightarrow \quad u \ge s \quad \forall s \in S$ which is (i).

By (ii), $u \le v$ for any upper bd. v of S. (*)

Fix $\varepsilon > 0$, but arbitrary.

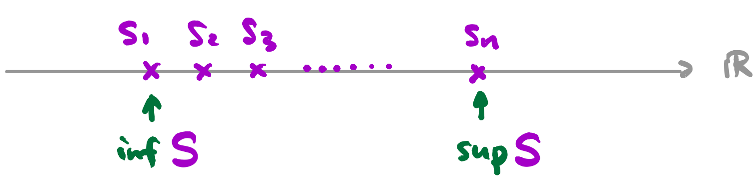
Since $u - \varepsilon < u$, (*) $\Rightarrow$ $u - \varepsilon$ cannot be an upper bd.

So, $\exists s' \in S$ s.t. $u - \varepsilon < s'$

"$\Leftarrow$" Exercise. $\square$

Examples:

1) $S = \{s_1, \dots, s_n\}$ "finite set" (Assume: $s_1 < s_2 < \dots < s_n$)

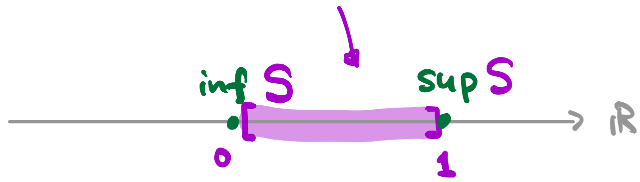


$\sup S = s_n$

$\inf S = s_1$

(Ex: Prove.)

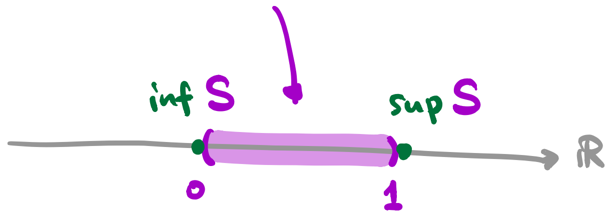
2) $S = [0, 1]$



$\sup S = 1 \in S$

$\inf S = 0 \in S$

3) $S = (0, 1)$



$\sup S = 1 \notin S$

$\inf S = 0 \notin S$